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Compactified Ruijsenaars-Schneider Systems

Motivation

Fehér and Kluck's recent discovery of new ***compactified*** trigonometric Ruijsenaars-Schneider models. These models describe N interacting particles moving on a circle. Their dynamics is governed by the ***Hamiltonian function***

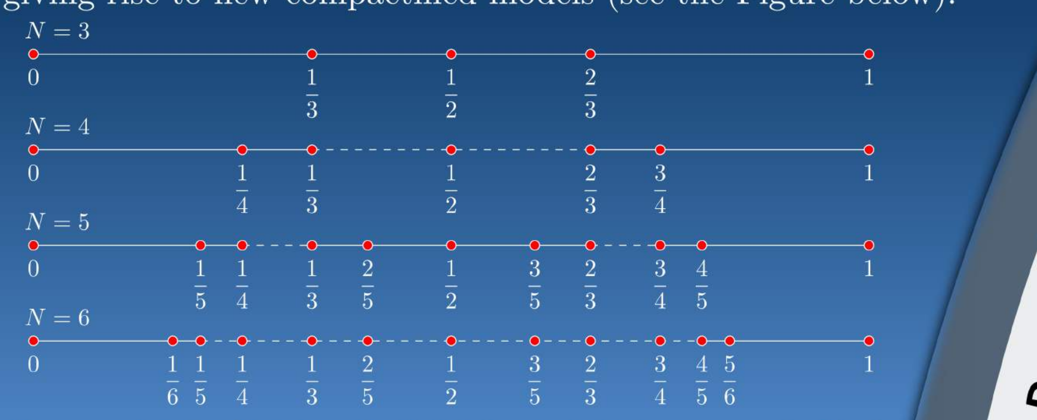
$$H(\boldsymbol{x},\boldsymbol{p}) = \sum_{j=1}^{N} \cos(\beta p_j) \sqrt{\prod_{k\neq j}\left(1 - \frac{\sin^2\left(\frac{\alpha\beta g}{2}\right)}{\sin^2 \frac{\alpha}{2}(x_j - x_k)}\right)}$$

with generalised coordinates $\boldsymbol{x} = (x_1,\dots,x_N)$, conjugate momenta $\boldsymbol{p} = (p_1,\dots,p_N)$, real-valued scale parameters α, β, and a coupling parameter g. Without loss of generality, we can take

$$\alpha > 0, \qquad \beta > 0, \qquad 0 < g < \frac{2\pi}{\alpha\beta}.$$

These compactified models are obtained from the usual trigonometric RS model by Wick-rotating the parameter β, i.e. replacing the real β by the imaginary $\mathrm{i}\beta$ in H.
Remarks. (1) Ruijsenaars introduced a slightly different Hamiltonian, namely, the square root of each factor is taken in the products appearing in H.
(2) That model was quantised and solved for $N=2$ by Ruijsenaars, and for $N>2$ by van Diejen-Vinet. Both works assumed that $0 < g < 2\pi/\alpha\beta N$.

The above Hamiltonian allows for ***new regimes*** for the coupling g giving rise to new compactified models (see the Figure below).



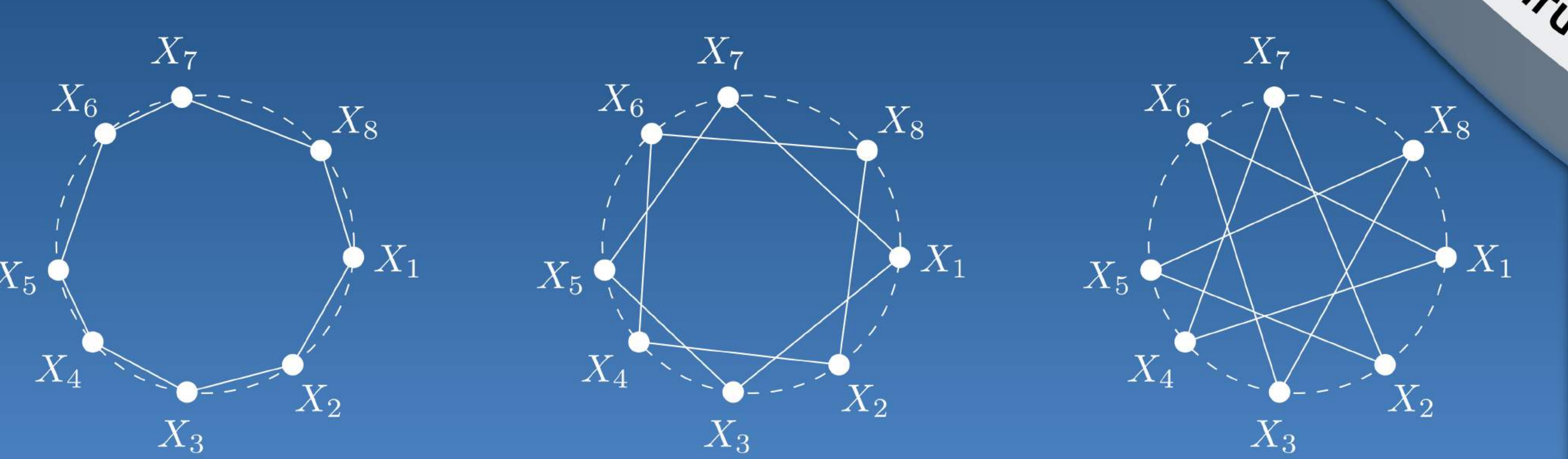
Range of $\alpha\beta g/2\pi$ for N=3,4,5,6. Red dots are excluded.
Intervals of type 1 (solid) and type 2 (dashed) g-values.

Abstract. The compactified Ruijsenaars-Schneider (RS) models are obtained from the standard RS systems (aka relativistic Calogero-Sutherland systems) by Wick rotation. Models with trigonometric and elliptic potentials describe particles moving on a circle with a chord distance-dependent pairwise interaction. Surprisingly, there are two dramatically different types of models (distinguished by the value of the coupling constant). For type 1 couplings particle distances have lower (and upper) bounds, while type 2 couplings allow particle collisions to happen resulting in a more complicated dynamics. The global phase space and quantisation of models with type 1 couplings are presented. Based on joint works with László Fehér and Martin Hallnäs.

Physical interpretation of the model

A possible interpretation of the model is that of N particles, moving on a circle of radius one half, positioned at $X_j = \frac{1}{2}e^{\mathrm{i}\alpha x_j}$, j=1,...,N with a pairwise interaction that depends on the square of the chord-distance.

Chords connecting p-nearest neighbours for N=8 with p=1,2,3, respectively. If the resulting graph (solid lines) is disconnected (such as the one in the middle), nearest neighbour particles can get arbitrarily close to each other regardless of lower/upper bounds on the drawn chord-distances. The number of graph components is gcd(N,p).



Classical reduced Hamitonians

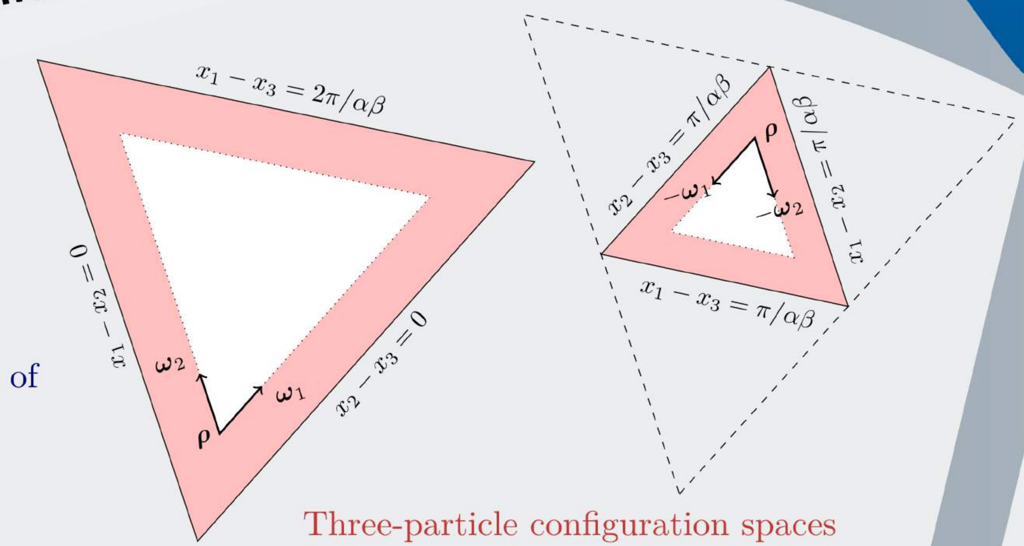
Consider the parameter

$$M = \frac{2\pi}{\alpha}p - \beta N g$$

The configuration space is the simplex consisting of points of the form:

$$\boldsymbol{x} = \boldsymbol{\rho}_p + \mathrm{sgn}(M)\textstyle\sum_{k=1}^{N-1} m_k \boldsymbol{\omega}_{k,p}$$

with $m_k > 0,\ \sum_{k=1}^{N-1} m_k < |M|$



Three-particle configuration spaces (white triangles with dotted sides) in the centre-of-mass plane with couplings yielding $M>0$ (left) and $M<0$ (right).

Integrability. A complete set of independent first integrals in involution:

$$H_r(\boldsymbol{x},\boldsymbol{p}) = \textstyle\sum_{\boldsymbol{\nu}\in S_N(\boldsymbol{\omega}_r)} \cos(\beta\langle\boldsymbol{\nu},\boldsymbol{p}\rangle)\sqrt{V_{\boldsymbol{\nu}}(\boldsymbol{x})V_{\boldsymbol{\nu}}(-\boldsymbol{x})}$$

where $V_{\boldsymbol{\nu}}(\boldsymbol{x}) = \prod_{\substack{\boldsymbol{a}\in A_{N-1}\\ \langle\boldsymbol{a},\boldsymbol{\nu}\rangle=1}} \frac{\sin\frac{\alpha}{2}(\langle\boldsymbol{a},\boldsymbol{x}\rangle+\beta g)}{\sin\frac{\alpha}{2}\langle\boldsymbol{a},\boldsymbol{x}\rangle}$

Root system notation

- Standard basis in $\mathbb{R}^N$: $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_N\}$
- Usual inner product on $\mathbb{R}^N$: $\langle\cdot,\cdot\rangle$
- $A_{N-1} = \{\boldsymbol{e}_j - \boldsymbol{e}_k \mid j,k = 1,\dots,N,\ j\neq k\}$
- A p-dependent base: $\boldsymbol{a}_{j,p} = \boldsymbol{e}_j - \boldsymbol{e}_{j+p}$
- Fundamental weights: $\boldsymbol{\omega}_{k,p}$: $\langle\boldsymbol{a}_{j,p},\boldsymbol{\omega}_{k,p}\rangle = \delta_{jk}$
- Root lattice Q, weight lattice Λ
- Cones: $Q_p^+ = \mathrm{span}_{\mathbb{N}_0}\{\boldsymbol{a}_{j,p}\}$, $\Lambda_p^+ = \mathrm{span}_{\mathbb{N}_0}\{\boldsymbol{\omega}_{k,p}\}$
- Dominance order: $\boldsymbol{\mu} \preceq \boldsymbol{\lambda}$ iff $\boldsymbol{\lambda} - \boldsymbol{\mu} \in Q_p^+$
- $\boldsymbol{\rho}_p = \beta g \sum_{k=1}^{N-1}\boldsymbol{\omega}_{k,p} - \frac{2\pi}{\alpha}\sum_{\ell=1}^{p-1}\boldsymbol{\omega}_{N-\ell,p}$

Main result

Quantization and exact solution of new compactified trigonometric Ruijsenaars-Schneider systems.

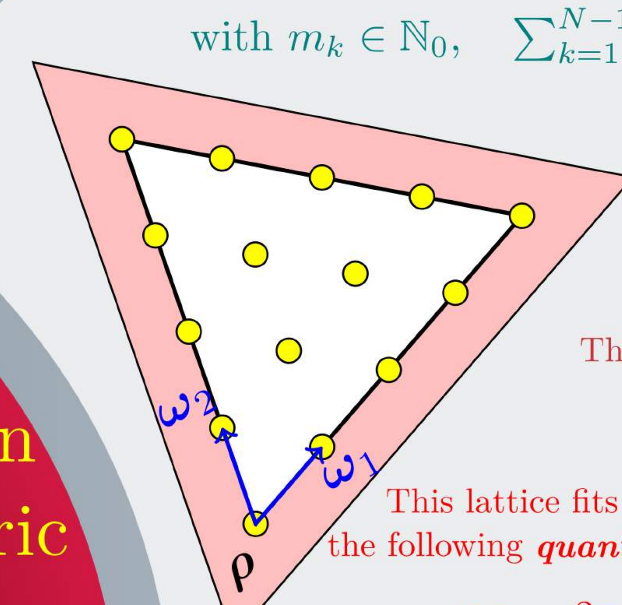
$$[\hat{H}_{r,M}, \hat{H}_{s,M}] = 0$$
$$\Downarrow$$
$$\hat{H}_{r,M}\Psi_{\boldsymbol{y},p} = E_r(\boldsymbol{y})\Psi_{\boldsymbol{y},p}$$
$$\Downarrow$$
$$E_r(\boldsymbol{y}) = \sum_{\boldsymbol{\nu}\in S_n(\boldsymbol{\omega}_r)} \cos\alpha\langle\boldsymbol{\nu},\boldsymbol{y}\rangle$$

The finite-dimenional Hilbert space

Consider the uniform lattice $\Lambda_{p,M}$ consisting of points

$$\boldsymbol{x} = \boldsymbol{\rho}_p + \mathrm{sgn}(M)\textstyle\sum_{k=1}^{N-1} m_k\boldsymbol{\omega}_{k,p}$$

with $m_k \in \mathbb{N}_0, \quad \sum_{k=1}^{N-1} m_k \leq |M|$



The 3-particle lattice $\Lambda_{1,4}$

This lattice fits the classical configuration space iff the following ***quantisation condition*** is satisfied:

$$M = \frac{2\pi}{\alpha}p - \beta N g \in \mathbb{Z}\setminus\{0\}$$

Let $L^2(\Lambda_{p,M})$ denote the finite-dimensional vector space of lattice functions $\phi\colon \Lambda_{p,M}\to\mathbb{C}$, equipped with the inner product

$$(\phi,\psi)_{p,M} = \textstyle\sum_{\boldsymbol{x}\in\Lambda_{p,M}} \phi(\boldsymbol{x})\overline{\psi(\boldsymbol{x})}$$

Its dimension equals the cardinality of $\Lambda_{p,M}$, which is

$$\binom{N-1+|M|}{|M|}$$

Quantum Hamiltonians

The following difference operators commute [Ruijsenaars '87]:

$$\hat{\mathcal{H}}_r = \textstyle\sum_{\boldsymbol{\nu}\in S_N(\boldsymbol{\omega}_r)} V_{\boldsymbol{\nu}}^{1/2}(\boldsymbol{x})\hat{T}_{\boldsymbol{\nu}}V_{\boldsymbol{\nu}}^{1/2}(-\boldsymbol{x}),\ r = 1,\dots,N-1$$

where $\hat{T}_{\boldsymbol{\nu}} = \exp(\langle\boldsymbol{\nu},\partial/\partial\boldsymbol{x}\rangle)$ is the translation operator acting on ϕ as

$$(\hat{T}_{\boldsymbol{\nu}}\phi)(\boldsymbol{x}) = \phi(\boldsymbol{x}+\boldsymbol{\nu})$$

Let us introduce the operators

$$\hat{\mathcal{H}}_{r,M} = \textstyle\sum_{\boldsymbol{\nu}\in S_N(\boldsymbol{\omega}_r)} V_{\boldsymbol{\nu}}^{1/2}(\boldsymbol{x})\hat{T}_{\mathrm{sgn}(M)\boldsymbol{\nu}}V_{\boldsymbol{\nu}}^{1/2}(-\boldsymbol{x})$$

Proposition.
(1) $\hat{\mathcal{H}}_{N-r,M}$ is the formal adjoint of $\hat{\mathcal{H}}_{r,M}$.
(2) The operators

$$\hat{H}_{r,M} = \tfrac{1}{2}(\hat{\mathcal{H}}_{r,M} + \hat{\mathcal{H}}_{N-r,M}),\ r = 1,\dots,N-1$$

are well-defined and self-adjoint on the Hilbert space $L^2(\Lambda_{p,M})$.

Spectra and joint eigenfunctions

Joint eigenfunctions

$$\Psi_{\boldsymbol{y},p}(\boldsymbol{x}) = \frac{1}{\mathcal{N}_0^{1/2}}\Delta_p(\boldsymbol{x})^{1/2}\Delta_p(\boldsymbol{y})^{1/2}P_{\sigma_p(\boldsymbol{y})}(\tilde{\boldsymbol{x}})$$

where $P_{\boldsymbol{\lambda}}$ denote the self-dual A_{N-1} Macdonald polynomials with parameters $t = e^{\mathrm{i}\alpha\,\mathrm{sgn}(M)g}$, $q = e^{\mathrm{i}\alpha}$.

A factorised joint eigenfunction

Consider the lattice function $\Delta_p\colon \Lambda_{p,M}\to\mathbb{R}$ given by

$$\Delta_p(\boldsymbol{x}) = \prod_{\boldsymbol{a}\in A_{N-1,p}^+} \frac{\sin\frac{\alpha}{2}\langle\boldsymbol{a},\boldsymbol{x}\rangle}{\sin\frac{\alpha}{2}\langle\boldsymbol{a},\boldsymbol{\rho}_p\rangle}\frac{(\langle\boldsymbol{a},\boldsymbol{\rho}_p\rangle + \mathrm{sgn}(M)g : \sin_\alpha)_{\langle\boldsymbol{a},\boldsymbol{x}-\boldsymbol{\rho}_p\rangle}}{(\langle\boldsymbol{a},\boldsymbol{\rho}_p\rangle + 1 - \mathrm{sgn}(M)g : \sin_\alpha)_{\langle\boldsymbol{a},\boldsymbol{x}-\boldsymbol{\rho}_p\rangle}}$$

where $(z:\sin_\alpha)_m$ stands for the trigonometric Pochhammer symbol

$$(z:\sin_\alpha)_m = \begin{cases} 1, & \text{if } m = 0,\\ \sin\frac{\alpha}{2}(z)\dots\sin\frac{\alpha}{2}(z+m-1), & \text{if } m = 1,2,\dots\\ \dfrac{1}{\sin\frac{\alpha}{2}(z-1)\dots\sin\frac{\alpha}{2}(z+m)}, & \text{if } m = -1,-2,\dots\end{cases}$$

Recurrence relations. For any $\boldsymbol{x}\in\Lambda_{p,M}$ and $\boldsymbol{\nu}\in S_N(\boldsymbol{\omega}_r)$ $r = 1,\dots,N-1$ satisfying $\boldsymbol{x} + \mathrm{sgn}(M)\boldsymbol{\nu}\in\Lambda_{p,M}$, we have

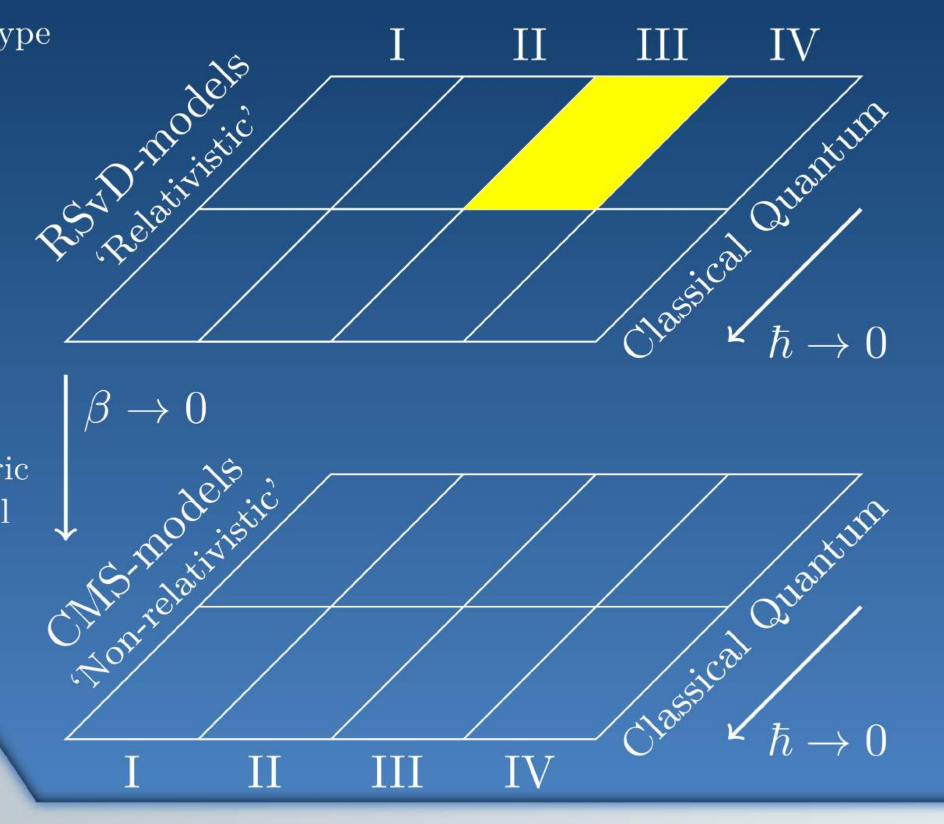
$$\frac{\Delta_p(\boldsymbol{x}+\mathrm{sgn}(M)\boldsymbol{\nu})}{\Delta_p(\boldsymbol{x})} = \frac{V_{\boldsymbol{\nu}}(\boldsymbol{x})}{V_{\boldsymbol{\nu}}(-\boldsymbol{x}-\mathrm{sgn}(M)\boldsymbol{\nu})}$$

Corollary. $\Delta_p(\boldsymbol{x})^{1/2}$ is a joint eigenfunction of the quantum Hamiltonians $\hat{H}_{r,M}$.

Calogero-Sutherland & Ruijsenaars-Schneider models

Roman numerals label the type of particle interaction:
I. Rational
II. Hyperbolic
III. Trigonometric
IV. Elliptic

Yellow field = Quantum Trigonometric RS model



Future directions

compactified models attached to root systems other than A_{N-1}

the case of type 2 coupling parameters (in progress)

finite-dimensional representations of $\mathrm{SL}(2,\mathbb{Z})$ using DAHAs

connecting new models to field theories

new quantum elliptic models?

Summary

We introduced new compactified trigonometric RS models with type 1 coupling parameters by

+ defining the appropriate quantum Hamiltonians as difference operators acting on a finite-dimensional Hilbert space of lattice functions,

+ providing an explicit solution to the corresponding eigenvalue problem in terms of Macdonald polynomials.

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